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operation by which it was obtained—formal division of the numerator by the denominator—cannot be performed when $x = 1$, although it can be performed for every other value of x.

Similarly, when we set out to "evaluate the indeterminate form $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ for $x = 1$," we must not think that we shall by sharpness of vision pierce an algebraic mist which hides from us the "true value" 1 of this expression for $x = 1$. We should mean that we are seeking the limit (if any exists) approached by $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ as x approaches 1; and we should mean nothing further than this.

For the value $x = 1$, $(x^2 - 1)/(x - 1)$ does assume the form $0/0$, and $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ may be said to assume the form ∞/∞; but neither fraction has any *value* for the value of x in question, nor can we by any amount of ambiguous argument or inaccurate analogy force it to have such a value.

ON THE GEOMETRICAL REPRESENTATION OF INDETERMINATE FORMS.

By R. E. MORITZ, University of Washington.

Every teacher of the calculus will have witnessed the difficulty beginners have in properly interpreting the so-called indeterminate forms $0/0$, ∞/∞, $0\cdot\infty$, $\infty - \infty$, etc. This should cause no surprise; for the meaning of the symbols 0 and ∞, even when taken by themselves, is not too clear to the average student and the difficulty is, of course, accentuated when these symbols are combined as in the indeterminate forms.

It should be remembered in this connection that the indeterminate forms have not always been correctly interpreted even by mathematicians and that some of them remained subjects of controversy until comparatively recent times. The first complete evaluation of the form $0/0$ dates from John Bernoulli[1] (1704) and it was not until 1755 that Euler[2] gave the first published treatment of the forms ∞/∞, $0\cdot\infty$, and $\infty - \infty$. But the true meaning of 0^0 eluded even the great Euler, for he arrived at the conclusion[3] that 0^0 must always equal 1 by a process of reasoning which would show equally well that $0/0$ must always equal 1. Euler's erroneous conclusion was shared by many subsequent mathematicians for three quarters of a century. New proofs (!) of his conclusion appeared in creditable journals by eminent mathematicians as late as the middle third of the nineteenth century,[4] and this notwithstanding the great weight of the authority of Cauchy, who, in his *Cours d'Analyse*,[5] had given the interpretation of the exponential indeterminate forms which is universally accepted today.

In view of the fact that modern textbooks on the calculus generally avail themselves of geometrical illustrations to introduce new or difficult concepts, it seems unfortunate that such aids are not used in introducing the concept of indeterminate forms. In this connection geometrical illustrations are so conspicuously absent that the cynically inclined reader might suspect writers of a conspiracy to mystify rather than to clarify the subject under discussion. With one notable exception[1] in the case of a text long out of use (and now out of print) the writer does not know of a single textbook in either English, French, or German, which so much as hints at the existence of geometrical illustrations in treating the subject of indeterminate forms, although every modern text, so far as the writer is aware, freely uses geometrical aids in introducing the concepts of the derivative and the definite integral.

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[1] *Opera*, vol. 1, pp. 401–405.

[2] *Opera Omnia*, series I, vol. 10, 1913, p. 576.

[3] Euler, *Opera Omnia*, series I, vol. 1, 1911, p. 65.

[4] Libri, G., Crelle's *Journal*, vol. 10, 1833, p. 303; Moebius, A. F., Crelle's *Journal*, vol. 12, 1834, p. 134.

[5] *Oeuvres*, 2d series, vol. 3, p. 70.

Lovitt,[2] in a short paper on the subject, gave five illustrations of indeterminate forms chosen from the fields of solid geometry, trigonometry, the geometry of the triangle, the geometry of the circle, and hydraulics respectively. These illustrations lack, however, the desired perspicuity and, in the majority of cases, are faulty in two respects: first, in that the variables actually reach their limits, thus rendering the expressions which give rise to the indeterminate forms meaningless; and second, in that the ratio, product, or difference which assumes the indeterminate form is constant and thus fails to illustrate the really important case of a function of x which approaches a limiting value but an indeterminate form as the variable approaches some definite limit.

It is the purpose of the present paper to exhibit some simple plane-geometric illustrations of the elementary indeterminate forms. Care has been exercised that in each case the function which in the limit assumes an indeterminate form is really a function of the variable assuming different values and not a constant in disguise. Moreover, the constructions are so conditioned that the question of dividing zero by zero, or infinity by infinity, cannot arise since the quantities considered are never zero or infinite. All our ratios are those of finite quantities which may be increased or decreased indefinitely but which, from the nature of their generation, can never be actually zero or infinite. The point here is that the problem of indeterminate forms may be so conceived that the question of dividing zero by zero, or infinity by infinity, which is the real source of confusion, need not arise.

Each of the figures that follow is so chosen that it illustrates at the same time each of the possible values zero, finite, infinite, which the indeterminate form under consideration may take. The last figure illustrates all the elementary indeterminate forms at once for the whole range of values which any of these forms may have. This figure is remarkable in that it interrelates one each of the four possible types of triangles, the ultimate values of whose sides are respectively

$$0, 0, 0; \qquad 0, a, b; \qquad a, \infty, \infty; \qquad \infty, \infty, \infty.$$

It would be interesting to know whether it is possible to construct a simpler figure that will serve the same purpose equally well or better.

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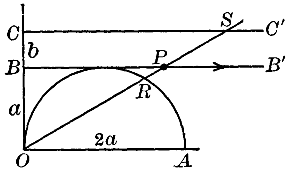
[1] DeMorgan, A., *Differential and Integral Calculus*, Introductory, p. 19.

[2] AMERICAN MATHEMATICAL MONTHLY, *1916*, 41.

Definitions. A variable distance is said to approach 0 as a limit, if it ultimately becomes, and remains, less than any preassigned distance however small.

A variable distance is said to approach a constant distance a as a limit if the difference between the constant distance and the variable distance ultimately becomes, and remains, less than any preassigned distance however small.

A variable distance is said to become infinite if it ultimately becomes, and remains, greater than any preassigned distance however large.



1. $\infty - \infty$. Consider the semicircle having $OA = 2a$ as a diameter. BB' is a tangent to the semicircle drawn parallel to OA, and CC' a line drawn parallel to BB' at a distance b from it on the side opposite from 0.

Let P be a point on BB' which moves indefinitely in the direction BB' at a uniform velocity. Join O to P and produce OP to meet CC' at S.

Now it is obvious that each of the variable distances OP, BP, RP, OS, ultimately becomes, and remains, larger than any preassigned distance however large, so that ultimately

$$OP - BP = \infty - \infty, \qquad OP - RP = \infty - \infty, \qquad OS - OP = \infty - \infty.$$

But

$$OP - BP = \frac{a^2}{OP + BP}, \text{ which approaches 0 as a limit,}$$

$$OP - RP = OR, \text{ which approaches } OA = 2a \text{ as a limit,}$$

$$OS - OP = PS, \text{ which becomes infinite;}$$

hence

$$OP \to \infty, \qquad BP \to \infty, \qquad RP \to \infty, \qquad OS \to \infty;$$

$$OP - BP \to 0, \qquad OP - RP \to 2a, \qquad OS - OP \to \infty.$$

If we take O as origin of coördinates, OA and OB as the directions of coördinate axes, and denote the coördinates of the point P by (x, a), we find

$$OP = \sqrt{x^2 + a^2}, \quad BP = x, \quad RP = (x - a)^2/\sqrt{x^2 + a^2}, \quad OS = \frac{a + b}{a}\sqrt{x^2 + a^2},$$

and as x becomes infinite we have

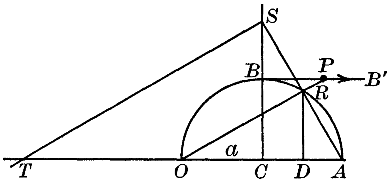
$$OP - BP = \sqrt{x^2 + a^2} - x = a^2/[\sqrt{x^2 + a^2} + x] \to 0,$$

$$OP - RP = \sqrt{x^2 + a^2} - (x - a)^2/\sqrt{x^2 + a^2} = 2ax/\sqrt{x^2 + a^2} \to 2a,$$

$$OS - OP = \frac{a + b}{a}\sqrt{x^2 + a^2} - \sqrt{x^2 + a^2} = \frac{b}{a}\sqrt{x^2 + a^2} \to \infty,$$

which verifies the preceding results analytically.

2. 0/0 and ∞/∞. Consider the semicircle having $OA = 2a$ as a diameter. BB' is the tangent to the semicircle drawn parallel to OA, P a point on it which moves indefinitely in the direction BB' at a uniform velocity.



Join P to O and denote the point in which OP intersects the semicircle by R. Join A and R and denote the point in which AR produced intersects the perpendicular CB to OA through the center C of the circle by S. Through S draw a perpendicular to AS and let this perpendicular intersect AO produced in T. Through R draw RD perpendicular to OA.

Now it is obvious that as P moves indefinitely along BB' in the direction of BB' each of the variable distances DA, AR, and RD ultimately becomes, and remains, less than any preassigned distance however small, and each of the variable distances CS, ST, and TC ultimately becomes, and remains, greater than any preassigned distance, however large, so that ultimately

$$\frac{DA}{RD} = \frac{0}{0}, \quad \frac{AR}{RD} = \frac{0}{0}, \quad \frac{RD}{DA} = \frac{0}{0}; \quad \frac{CS}{TC} = \frac{\infty}{\infty}, \quad \frac{ST}{TC} = \frac{\infty}{\infty}, \quad \frac{TC}{CS} = \frac{\infty}{\infty}.$$

But the triangles RDA, SCA, ODR, and TCS are similar, so that

$$\frac{DA}{RD} = \frac{CS}{TC} = \frac{DR}{OD}, \text{ which approaches 0 as a limit,}$$

$$\frac{AR}{RD} = \frac{ST}{TC} = \frac{RO}{OD}, \text{ which approaches 1 as a limit,}$$

$$\frac{RD}{DA} = \frac{TC}{CS} = \frac{SC}{CA}, \text{ which becomes infinite;}$$

hence

$$DA \rightarrow 0, \quad RD \rightarrow 0, \quad AR \rightarrow 0, \quad CS \rightarrow \infty, \quad TC \rightarrow \infty, \quad ST \rightarrow \infty;$$

$$\frac{DA}{RD} \rightarrow 0, \quad \frac{CS}{TC} \rightarrow 0, \tag{1}$$

$$\frac{AR}{RD} \rightarrow 1, \quad \frac{ST}{TC} \rightarrow 1, \tag{2}$$

$$\frac{RD}{DA} \rightarrow \infty, \quad \frac{TC}{CS} \rightarrow \infty. \tag{3}$$

We may obtain the same results analytically as follows. Take O as origin and OA as the direction of the x-axis of coördinates. Denote the coördinates of the point P by (x, a), then we find

$$DA = 2a^3/(x^2 + a^2), \quad AR = 2a^2/(\sqrt{x^2 + a^2}), \quad RD = 2a^2x/(x^2 + a^2),$$

$$CS = x, \quad ST = x\sqrt{x^2 + a^2}/a, \quad TC = x^2/a.$$

Substituting these values in the left members of (1), (2), (3), we have as x becomes infinite

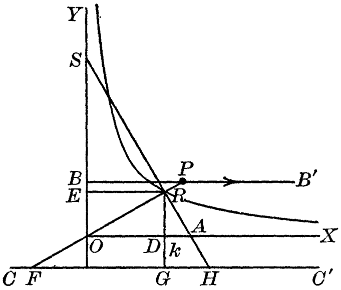
$$\frac{DA}{RD} = \frac{2a^3/(x^2+a^2)}{2a^2x/(x^2+a^2)} = \frac{a}{x} \to 0, \qquad \frac{CS}{TC} = \frac{x}{x^2/a} = \frac{a}{x} \to 0;$$

$$\frac{AR}{RD} = \frac{2a^2/(\sqrt{x^2+a^2})}{2a^2x/(x^2+a^2)} = \frac{\sqrt{x^2+a^2}}{x} \to 1, \qquad \frac{ST}{TC} = \frac{x\sqrt{x^2+a^2}/a}{x^2/a} = \frac{\sqrt{x^2+a^2}}{x} \to 1;$$

$$\frac{RD}{DA} = \frac{2a^2x/(x^2+a^2)}{2a^3/(x^2+a^2)} = \frac{x}{a} \to \infty, \qquad \frac{TC}{CS} = \frac{x^2/a}{x} = \frac{x}{a} \to \infty.$$

3. $\infty \cdot 0$. Consider a branch of the equilateral hyperbola with $\sqrt{2}$ for the semi-major axis. Let O represent the center and OX, OY, the asymptotes to the hyperbola. Draw two parallels, BB' and CC', to one of the asymptotes on opposite sides of it and at distances a and k respectively.

Let P be a point on BB' which moves indefinitely in the direction of BB' at a uniform velocity. Join OP and denote its intersection with the hyperbola by R and its intersection with CC' by F. Through R draw a perpendicular to OR and denote its intersection points with OY, OX, and CC', by S, A, and H respectively. From R draw a perpendicular to OX, intersecting OX and CC' in D and G respectively. Through R draw a parallel to OX intersecting OY at a point E.



As the point P moves indefinitely in the direction BB' it is obvious that each of the variable distances FG, OD, ER, ES, ultimately becomes, and remains, greater than any preassigned distance however great, and each of the variable distances OE, DR, DA, GH, ultimately becomes, and remains, less than any preassigned distance however small, while GR approaches $GD = k$ as a limit; hence ultimately

$$OD \cdot DA = \infty \cdot 0, \qquad FG \cdot GH = \infty \cdot 0, \qquad ES \cdot OE = \infty \cdot 0.$$

But from the pairs of similar triangles ODR, RDA, FGR, RGH, SER, REO, we have

$$OD \cdot DA = (DR)^2, \text{ which approaches 0 as a limit,}$$

$$FG \cdot GH = (GR)^2, \text{ which approaches } k^2 \text{ as a limit,}$$

$$ES \cdot OE = (ER)^2, \text{ which becomes infinite;}$$

hence

$$OD \to \infty, \quad FG \to \infty, \quad ES \to \infty; \quad DA \to 0, \quad GH \to 0, \quad OE \to 0;$$

$$OD \cdot DA \to 0, \tag{4}$$

$$FG \cdot GH \to k^2, \tag{5}$$

$$ES \cdot OE \to \infty. \tag{6}$$

To verify these results analytically, take O as the origin, OX as the x-axis of coördinates. Then, if we denote the coördinates of the point P by (x, a), we find

$$OD = \sqrt{(x/a)}, \qquad FG = (x/a)[k + \sqrt{(a/x)}], \qquad ES = \sqrt{(x/a)^3},$$

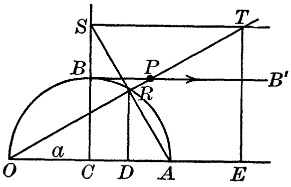
$$DA = \sqrt{(a/x)^3}, \qquad GH = (a/x)[k + \sqrt{(a/x)}], \qquad OE = \sqrt{(a/x)}.$$

On substituting these values in the left members of (4), (5), (6), we have as x becomes infinite,

$$OD \cdot DA = \sqrt{(x/a)} \cdot \sqrt{(a/x)^3} = a/x \to 0,$$

$$FG \cdot GH = (a/x)[k + \sqrt{(a/x)}] \cdot (x/a)[k + \sqrt{(a/x)}] = [k + \sqrt{(a/x)}]^2 \to k^2,$$

$$ES \cdot OE = \sqrt{(x/a)^3} \cdot \sqrt{(a/x)} = x/a \to \infty.$$



4. The figure illustrates all the preceding cases of indeterminate forms at the same time. The figure is like the one in § 2 with two additional lines, a line through S drawn parallel to OA and a perpendicular to OA through T, the point in which this line intersects OP produced.

As the point P moves indefinitely with a uniform velocity in the direction BB' the variable sides of the similar right triangles

$$RAD, \qquad OAR, \qquad SAC, \qquad TSR,$$

ultimately become or approach

$$0,\ 0,\ 0; \qquad 0,\ 2a,\ 2a; \qquad a,\ \infty,\ \infty; \qquad \infty,\ \infty,\ \infty.$$

We have then

$$AS - CS = \infty - \infty, \qquad OT - RT = \infty - \infty, \qquad OT - PT = \infty - \infty;$$

$$RS/ST = \infty/\infty, \qquad ST/RT = \infty/\infty, \qquad RT/RS = \infty/\infty;$$

$$CS \cdot DA = \infty \cdot 0, \qquad CS \cdot DR = \infty \cdot 0, \qquad RT \cdot AR = \infty \cdot 0;$$

$$DA/DR = 0/0, \qquad AR/DR = 0/0, \qquad DR/DA = 0/0.$$

Again it appears from the obvious geometrical relations that

$$AS - CS = (CA)^2/(AS + CS), \qquad RS/ST = AR/OA, \qquad CS \cdot DA = DR \cdot CA,$$

$$DA/DR = AR/OR,$$

the right member of each of which approaches 0 as a limit;

$$OT - RT = OR, \quad ST/RT = OA/OR, \quad CS \cdot DR = CA \cdot OD, \quad AR/DR = OA/OR,$$

and the right members of these approach the limits $2a$, 1, $2a^2$, 1, respectively;

finally

$$OT - PT = OP, \quad RT/RS = CS/CA, \quad RT \cdot AR = RS \cdot OR, \quad DR/DA = CS/CA,$$

the right member of each of which becomes infinite.

We have therefore as the limits of the ratios involved

$$\begin{array}{rrr} AS - CS \to 0, & OT - RT \to 2a, & OT - PT \to \infty; \\ RS/ST \to 0, & ST/RT \to 1, & RT/RS \to \infty; \\ CS \cdot DA \to 0, & CS \cdot DR \to 2a^2, & RT \cdot AR \to \infty; \\ DA/DR \to 0, & AR/DR \to 1, & DR/DA \to \infty. \end{array}$$

RECENT PUBLICATIONS.

REVIEWS.

Elementary Calculus. By W. F. Osgood. New York, The Macmillan Company, 1921. 12mo. 10 + 224 pages. Price $2.40.

Preface: "The object of this book is to present the elements of the Differential Calculus in a form easily accessible for the undergraduate. It is possible, from the very beginning, to illustrate the ideas and methods of the Calculus by means of applications to physics and geometry, which the student can readily grasp, and which will seem to him of interest and value. To do this, the stress in the illustrative examples worked in the text must be laid first of all on the thought which underlies the method of solution, in distinction from the exposition of a process, reduced in the worst teaching to rules, whereby the answer can be obtained. The treatment of maxima and minima, Chapter III, §§ 2, 3, and curve tracing, Chapter III, § 5, and Chapter VII, § 10, will serve to show what is here meant.

"It is, however, also essential that the student receive thorough training in the formal processes and the technique of the Calculus, and this side has been treated with care and completeness. Note, for example, the differentiation of composite functions in Chapter II, § 8, and the exposition of the use of differentials in differentiating in Chapter IV, §§ 4, 5.

"An important application of the graphical methods, with which the Calculus is so intimately associated, is that of solving approximately numerical equations which do not come under the standard rules of algebra and trigonometry. Hitherto, however, little attempt has been made to present this subject, simple as it is, in any systematic and elementary manner. In Chapter VII the common methods in use by physicists and others who apply the Calculus are set forth and illustrated by simple examples.

"The book might have included a brief treatment of curvature and evolutes, and the cycloid. But probably most teachers of the Calculus will prefer to take up integration next, and so the closing chapter is devoted to the last of the elementary functions, the inverse trigonometric functions, with special reference to their one great application in the elements of mathematics, namely, their application to integration.

"The book is so written that it can be adapted, if desired, to an abridged course, in which, after the fundamentals of the first three chapters have been covered, any of the remaining topics can be treated briefly, and thus a wide scope in subject matter is possible, even when the time is short."

Contents—Chapter I: Introduction, 1–12; II: Differentiation of algebraic functions. General theorems, 13–45; III: Applications, 46–80; IV: Infinitesimals and differentials, 81–104; V: Trigonometric functions, 105–145; VI: Logarithms and exponentials, 146–165; VII: Applications, 166–205; VIII: The inverse trigonometric functions, 206–224.